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RELAXATION OF PROBLEMS OF OPTIMAL STRUCTURAL DESIGN

A. V. CHERKAEV
Department of Mathematics, The University of Utah, Salt Lake City, UT 84112, U.S.A.

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Abstract—The problem of optimal structural design is considered. The goal of design is to minimize an integral functional which characterized the quality of construction. Classes of optimal microstructures of composites made of given materials are described and analysed. The proposed method deals with the averaged description of minimizing sequences of materials layout which physically are the composites with a special microstructure. In contrast with the known approach of the G-closure description, the suggested procedure does not require the description of the full variety of effective properties of an arbitrary composite, but instead selects the class of structures which are the only candidates to optimal design. The approach is illustrated by the structural design problems of conducting and elastic composites.

1. INTRODUCTION

We describe a method of solution of optimal structural design problems for conducting or elastic inhomogeneous bodies. The problem is to find the optimal layout of several materials with different material constants throughout the construction; the boundary conditions and the loading are supposed to be known. The goal of the project is to minimize a lower weakly semi-continuous functional of the solution of the corresponding boundary value problem of conductivity or elasticity, that is, of the thermal or deflection field. The lower weak semi-continuity of the functional basically means that its value does not change if the original problem of the displacement of given materials is replaced with the problem of the displacement of materials with microstructures or composites assembled from them (see, e.g. Dagorogna, 1982). The physical properties of a composite are presented by an effective properties tensor which depends on the structure of a composite. The problem is to determine that structure which "adopts" the construction to the functional and to the loading/boundary conditions.

Examples of problems of this kind for elastic bodies (loaded with a fixed load) include:

—minimization of the norm of the displacement (or of the magnitude of the vibration) in some point or in some region on the boundary of the body. Minimization of the integral of a component of the displacement through some given region.

—maximization of the "sensitivity" of a construction which is the norm of the displacement in some regions of the body.

—minimization of the norm of the difference between the displacement field caused by a loading and a field we would like to achieve.

Similar problems arise in heat or electricity conducting bodies assembled from different materials. For example it may be required:

—to minimize a norm of difference between the existing temperature field and some fixed field we would like to have.

—to minimize the thermal energy within some region of the construction.

Study of these problems has been in progress for more than a decade and several approaches have been developed. The most investigated class of optimal design problems of this kind is the problem of minimization of the energy stored in an inhomogeneous material. Physically, minimization of the energy means the maximization of overall strength of the construction. The engineering justifications of it have been discussed many times, see, for example, Rozvany (1989) or Kirsch (1989); mathematically it is a simple optimal design problem. Indeed, the problem can be formulated as a variational problem of minimizing the energy with respect to both stresses (strains) and displacements of the materials (design variables); this allows one to apply the known and the recently developed methods of calculus of variations.

It was found about two decades ago that this problem does not have a classical (smooth) solution (Olhoff, 1974; Lurie and Cherkaev, 1976; Cheng and Olhoff, 1981) and needs a regularization procedure. In Kohn and Strang (1986) it was pointed out that problems of this kind are ill-posed because of non-convexity of the functional with respect to design variables; solutions of such problems of optimal layout of materials do not exist. The nonexistence simply means that the optimal layout necessarily includes infinitely small regions occupied with initially given materials; these regions are mixed in a special way to form optimal microstructures.

The solutions of such problems require special relaxation methods. Namely, the problem of a body with minimal stored energy can be divided into two parts. The first one, the local problem, asks for the periodic structure of a composite which corresponds to minimum energy stored in it under uniform loading. It has been shown (Kohn and Strang, 1986; Tartar, 1985; Lurie and Cherkaev, 1984, 1986; Francfort and Murat, 1987, 1991; Gibiansky and Cherkaev, 1986, 1987, 1988; Avellaneda, 1987; Milton, 1990a) that the local problem possesses, in many cases, an analytical solution which is given by so-called quasi-convex envelope (Kohn and Strang, 1986) of the functional. The points of the envelope represent the energy of optimally designed microstructures and these microstructures picture minimizing sequences of layout of materials.

The second part, the global problem, asks for the disposition of optimal microstructures throughout a body; usually it must be solved numerically. By doing this, the optimal microstructures are replaced by the optimal homogenized anisotropic materials. These properties are expressed by the effective properties tensors; the effective tensor smoothly varies from point to point following the changes of the stress (or current) fields. The last problem can be treated as the problem of the state of nonlinear conducting or elastic material whose energy function has been found analytically as the energy of optimal microstructures.

This procedure guarantees the existence of an optimal solution in the enlarged class of control which includes the original materials and composites made of them. The arising problems of stability of this solution are discussed, for example, in Cherkaev (1992); Rozvany *et al.* (1993).

The most difficult and novel part of the whole problem is the problem of building the quasiconvex envelope; several methods have been developed for it. We mention the translation method (Lurie and Cherkaev, 1984; Murat and Tartar, 1985a; Milton, 1990a,b; Cherkaev and Gibiansky, 1993a) and the estimates of Hashin–Shtrikman type (Hashin and Shtrikman, 1963; Milton, 1990a,b; Allaire and Kohn, 1993) which give the lower estimate of the energy stored in microstructures of arbitrary geometry. Special minimizing sequences (special microstructures) have been constructed in the papers mentioned above, and it has been proved that the energy reaches its minimum within these structures. The microstructures, called laminates of some rank, which will be described later, turn out to be of special importance; these structures have been investigated in many papers (see, e.g. Milton, 1980, 1986; Lurie and Cherkaev, 1981, 1984; Francfort and Murat, 1991; Avellaneda, 1987; Lipton, 1993; Gibiansky and Cherkaev, 1986).

The first solved problem of this kind is the design of an optimal inhomogeneous elastic bar with extremal rigidity (Lavrov *et al.*, 1980; Kohn and Strang, 1983). It was proved that only the properly oriented laminate materials must be used for both maximizing or minimizing the torsional rigidity of the bar. This conclusion also remains true for the bar made of non-linear elastic materials and of elastic–plastic ones (Gibiansky and Cherkaev, 1988). The problem of optimal design of thin plates, which store minimum elastic energy, was solved in the same way (Gibiansky and Cherkaev, 1986, 1987; Bendsoe and Kikuchi, 1988; Suzuki and Kikuchi, 1991; Bendsoe *et al.*, 1992; Allaire and Kohn, 1993; Jog *et al.*, 1993) as well as the problem of optimal elastic three-dimensional composites (Gibiansky and Cherkaev, 1987; Allaire and Kohn, 1993; and the related paper Francfort and Murat, 1991); it was found that the optimal structures here are the matrix lamination of second- and third-rank.

The more general class of weakly lower semicontinuous goal functionals discussed here also requires the relaxation procedure because of the same reason; the absence of solutions of the initial problems. Evidence of the absence of solutions of the problem and of the need of regularization procedure has been pointed out in early papers by Lurie (1970); Murat (1972, 1977); Tartar (1978). This time the problem is not a simple variational one, but it turns out to be a variational problem with differential restrictions which display the equation of state. The technique of relaxation of these problems is less developed. However, several approaches have been already suggested for relaxation of this kind of structural design problem. One approach (Lurie *et al.*, 1982) is based on the description of the so-called G-closure of the class of given material properties (Lurie and Cherkaev, 1981, 1984a,b, 1986; Francfort and Murat, 1987; Milton 1990b). The G-closure is by definition the set of all possible tensors of effective properties of all microstructures, assembled from the given materials. Clearly, if the complete description of the G-closure set is available, one can be sure that the appropriate element (the effective tensor of an optimal microstructure) belongs to it. Therefore, the reformulation of the initial problem by replacing the initial set of materials by its G-closure makes the problem well-posed. (Dealing with optimal design problems with restrictions on the amounts of given materials we may also need the description of the so-called G_m-closure (Lurie and Cherkaev, 1984) which is a set of all possible tensors of effective properties of all microstructures, assembled from given materials taken in given volume fractions.)

However, the G-closure problem itself is rather complicated and only a few results are attained now. Namely, the full description of G-closures and G_m-closures for a set of conducting media has been described in the papers by Lurie and Cherkaev (1981, 1984a,b, 1986); Murat and Tartar (1985b); Francfort and Murat (1987). A more complicated example of G_m-closure has been obtained in Cherkaev and Gibiansky (1992) where the problem has been solved for the electromagnetic dielectrics in two dimensions. In the aforementioned papers, an analytical procedure to determine the bounds of G-closures has been suggested. It requires solving a simple variational problem of minimization of the sum of densities stored in a mixture energy and/or complementary energy. It is supposed that the periodic composite structure has been placed under action of several orthogonal external periodic fields. The minimizers in the variational problem are both the fields and the materials distribution. The solution of a corresponding variational problem has been found in an explicit form, thus the bounds of G-closure have been described explicitly. The description of this approach can be found in many papers, for example, Lurie and Cherkaev (1984, 1986); Murat and Tartar (1986); Milton (1990a); Cherkaev and Gibiansky (1992, 1993). Note, that we do not mention here numerous papers devoted to the partial description of G-closures, such as the description of the variety of isotropic composites. The reason for it is that the mentioned approach to optimal design needs the knowledge of all the G-closure set because it is not known *a priori* what composite is the best one.

The explicit representation of the G-closure of conducting composites has been used in Gibiansky *et al.* (1988) for solving the optimal design problem of the thermo-lens, i.e. of a body which condenses the thermal current due to its inhomogeneity. In particular, it has been found that only pure materials and the most uncomplicated laminate microstructures must be used to assemble an optimally designed lens.

For elastic composites, however, the problem of G-closure is far from complete and there is little hope of obtaining results for the general case in reasonable time. Besides this, such a description consists of a large number of inequalities which make them difficult to work with. On the other hand, the description of the G-closure is only sufficient, but not necessary, for determination of the optimal structures, and we need essentially much less information for this goal. The major point is that one could *a priori* restrict oneself to a special class of microstructures which form an optimal solution of the correspondent variational problem. The problem of determination of this class is the main goal of the present paper; this problem turns out to be simpler than the G-closure problem.

We mention that more straightforward approaches to the relaxation of the optimal problems than the universal G-closure procedure have been suggested. Raitum (1981, 1983) noted that the given values of the pair of current and gradient fields in the low state do not determine the tensor of properties completely. One can consider the equivalence class of anisotropic tensors which produce the same current under the given gradient field. One could therefore restrict oneself to description of equivalence classes of effective tensors and to find a set of microstructures which represent each of them. This idea allowed Raitum to prove that only laminates need to be used for optimal conducting media.

In a recent paper, Lurie (1994) suggested a different approach based on direct estimates of the value of the min–max augmented functional that replaces the original optimization problem with differential constraints. He used lower and upper bounds for the augmented integrand. These bounds are generated, respectively, by a suitable laminar microstructure and a specially constructed modification of the original integrand he called the polysaddle envelope. When these bounds coincide, the problem becomes well posed. In Lurie (1994), the coincidence of bounds was demonstrated for a number of situations encountered in the plate problem. This technique has been suggested before (Lurie, 1990).

Here we develop a different method, which allows the optimal structures to be immediately determined. We simply reduce the problem to a regular, minimal variational problem by demonstration that any optimally designed conducting (elastic) composite structure corresponds to the minimum value of the sum of specific energy caused by an exterior uniform gradient field (stress) and a specific complementary energy caused by an exterior uniform current (strain) field. This gives a qualitative information of the types of optimal microstructures and makes the problem of optimal composites a specific part of the general G-closure problem. Namely, we indicate components of the boundary of G-closure, to which optimal composites should belong.

2. STATEMENT OF THE PROBLEM

2.1. *The problem*

We consider a problem of optimal design of a conducting or elastic body $\mathcal{O}$. The state of the body is described as a solution of a conductivity (elasticity) problem with an elliptic differential operator $L(D)$:

$$L(D)w = q \tag{1}$$

where $D = D(z)$ is the tensor of physical properties of the material placed in the point x, and q is an exterior loading. Some boundary conditions should be fixed on the boundary $\partial\mathcal{O}$ of the body $\mathcal{O}$. For simplicity, we will consider the case of uniform conditions:

$$w|_{\partial\mathcal{O}} = 0. \tag{2}$$

The operators $L(D)$ for the problems under consideration are equal either to the scalar operator $L'(D)$ (conductivity problem)

$$L'(D) = \nabla \cdot D(z) \cdot \nabla \tag{3}$$

where $C = \{c_{nmij}\}$ is a symmetric positive fourth-order tensor of elastic constants of the material in the point x; $(\cdot)$ means the scalar product; the solution $w(x)$ means the temperature; or the vector operator $L''(D)$ (elasticity problem)

$$L''(C) = \nabla : C(x) : (\nabla + \nabla^T) \tag{4}$$

where $C = \{c_{nmij}\}$ is a symmetric positive fourth-order tensor of elastic constants of the material; the solution w means the displacement vector. The symbol (:) denotes a double contraction of indices of tensors, for example, if $A = \{a_{nmij}\}$ is a fourth-rank tensor and $B = \{b_{ji}\}$ is a second-rank tensor, then:

$$A : B = \sum_{ij} a_{nmij} b_{ji}. \tag{5}$$

Note that both elliptic operators L' and L'' are self-adjoint: for any potentials α, β which vanish on the boundary $\partial\mathcal{O}$ the equality holds:

$$\int_{o} \beta L\alpha \, dx = \int_{\mathcal{O}} \alpha L\beta \, dx. \tag{6}$$

The optimal design problem we deal with is formulated in the following way: minimize the functional $I(w)$ of the solution w of the corresponding boundary value problem of conductivity (elasticity):

$$I(w) = \int_{\mathcal{O}} r \cdot w \, dx \tag{7}$$

where $r = r(x)$ is a given weight function. Additional restrictions expressing the limitations of the amounts of the given materials can be added to the problem in a standard way.

One can see that most of the design problems listed above in the Introduction permit the described formulation.

Remark 1. One can consider more general functionals of the form

$$I'(w) = \int_{\mathcal{O}} f(w, \nabla w) \, dx \tag{8}$$

where w is a scalar (in the conductivity problem) or vector (in the elasticity problem) potential. The function $f(\cdot, \cdot)$ is supposed to be convex with respect to the second argument (which does provide the lower semi-continuity of $I(w)$); see, e.g. Dagorogna (1982). It is easy to demonstrate that the results and the method of the present paper are valid for that problem as well.

2.2. *Augmented functional and properties of the minimizers*

We begin with the construction of the augmented functional I_A of the optimal problem by adding to the minimizing functional I the differential equation of the state (1) with a Lagrange multiplier $\lambda = \lambda(x)$.

$$I_A = \min_{D(x)} \min_{w(x)} \max_{\lambda(x)} \int_{\Omega} [r(x)w + \lambda(x)[L(D(x))w - q]] \, dx. \tag{9}$$

Applying the usual arguments we conclude that the cost of the augmented problem is equal to the value of the functional I.

The equation for the Lagrange multipliers in the conductivity problem may be found in the standard way by the variation of the augmented functional (9) with respect to w:

$$\delta I_A = R\delta w = 0 \tag{10}$$

which is implicated due to (6)

$$R = L'(D)\lambda - r = 0. \tag{11}$$

The boundary conditions for λ are:

$$\lambda|_{\partial\mathcal{O}} = 0. \tag{12}$$

Similarly, the Lagrange multiplier λ for the elasticity problem described by the vector-valued equation of state (4) is a vector and the representation (11) should be replaced by a vector-valued equation:

$$R = L''(D)\lambda - r = 0 \tag{13}$$

similar to the equation (1) of the state. This equation is to be solved with the corresponding homogeneous boundary conditions.

If, moreover, the minimizing functional I is equal to the work of external loading

$$r = q \tag{14}$$

and the boundary conditions for w and λ coincide (here they are homogeneous: $w = \lambda = 0$) then the problem for Lagrange multiplier (11), (14) coincides with the original one (1). Solutions of these problems coincide as well:

$$\lambda = w. \tag{15}$$

In this case we call the optimal problem a self-adjoint one. Also, we deal with a self-adjoint optimization problem when the problem for Lagrange multiplier differs only by sign from the original one. This occurs when the negative of the value of the work is minimized (or when the work is maximized):

$$r = -q \rightarrow \lambda = -w. \tag{16}$$

These self-adjoint problems ask for the conducting or elastic body storing minimal or maximal energy; this problem was being investigated by a number of authors as we have mentioned in the Introduction.

Otherwise, the optimization problem is non-self-adjoint but the operator $L(D)$ is still a self-adjoint one. Therefore, the Lagrange multiplier λ generally satisfies a boundary value problem for the same operator $L(D)$ but with different boundary conditions and right hand sides. Physically it means that λ may be considered as a field of the same nature as the field w as it corresponds to the same inhomogeneous media $D(x)$ but is caused by different external loading and boundary conditions (which generally may depend on the field w).

3. A LOCAL OPTIMIZATION PROBLEM

3.1. *Relaxation*

The classical precept for solving the extremal problems is the following: supposing the dependent variables (∇w) and Lagrange multipliers ($\nabla\lambda$) are smooth functions of the point x, one should express the controls ($D = D(x)$) through these variables by solving the system of necessary conditions of optimality. These conditions particularly tell us that D must deliver the minimum of the integrand of augmented functional (9) calculated with "frozen" dependent variables and Lagrange multipliers (see Lurie, 1993 for detailed references). Then one should use the obtained representation $D_{opt} = D_{opt}(\nabla w, \nabla\lambda)$, together with eqns (1) and (11), to determine the variables w, λ.

This procedure means that the operations of $\max_\lambda$ and $\min_D$ in the right hand side of (9) have been interchanged; the result generally presents a lower estimate of the augmented functional:

$$I_A \geqslant I_{A+} = \min_{D(x)} \min_{w(x)} \max_{\lambda(x)} \left\{ \int_\Omega [f(w, \nabla w) - q\lambda + \lambda(x) L(D(x)) w] \, dx \right\} \tag{17}$$

and must be supplemented by the demonstration of its attainability.

However, the solution of structural design problems cannot be obtained by following this construction directly because the initial assumption of smoothness of the dependent variables does not hold. On the contrary, it has been shown many times (see, e.g. Murat, 1972; Olhoff, 1974; Lurie and Cherkaev, 1976; Cheng and Olhoff, 1981; Lurie *et al.*, 1982; Murat and Tartar, 1985b; Kohn and Strang, 1986), using different types of arguments, that the variables as well as the control function are characterized by fine-scale oscillations. Therefore, the evaluation of the control parameters cannot be done algebraically but leads to a variational min–max problem. Namely, the algebraic procedure of solution of necessary conditions must be replaced by the solution of the so-called *local* problem, which is the variational problem of optimization of the structure in an infinitesimally small neighborhood of a point of the designed body.

The weak continuity of the problem ensures that the described program is correct: this property means that the value of the functional I_A (and the cost functional I) does not change if the fields are averaged throughout an infinitely small regular domain Ω; the average means

$$\langle (\cdot) \rangle = \frac{1}{\|\Omega\|} \int_\Omega (\cdot) \, dx \quad (\|\Omega\| \to 0) \tag{18}$$

where Ω is a rectangle which shrinks to its center. By averaging, we replace the original problem (9) with the problem

$$I_A = \min_{D(x)} \min_{\langle w(x) \rangle} \max_{\langle \lambda(x) \rangle} \int_{\mathcal{O}} (\langle wr \rangle + \langle q\lambda \rangle + \langle \lambda L(D) w \rangle) \, dx. \tag{19}$$

It is easy to calculate the first two terms of the integrand, supposing that functions $r(x)$ and $q(x)$ are sufficiently smooth almost everywhere:

$$\langle w \cdot r \rangle = \langle w \rangle \cdot r, \quad \langle q \cdot \lambda \rangle = q \cdot \langle \lambda \rangle \tag{20}$$

but the calculation of the last term is a difficult problem. This problem is in fact the main goal of the present paper. We want to calculate the average through Ω

$$B = \min_{D(x)} \min_{w} \max_{\lambda} \int_\Omega \lambda L(D) w \, dx \tag{21}$$

and to express the result in terms of the averaged fields ∇w and $\nabla \lambda$:

$$B = B(\langle \nabla \lambda \rangle, \langle \nabla w \rangle). \tag{22}$$

Doing this we divide the optimization problem into the local and global parts. The global problem has a form

$$I_R = \min_{\langle D(x) \rangle} \min_{\langle w(x) \rangle} \max_{\langle \lambda(x) \rangle} \int_{\mathcal{O}} (\langle w \rangle r + q \langle \lambda \rangle + B(\langle \nabla \lambda \rangle, \langle \nabla w \rangle)) \, dx; \tag{23}$$

it is called the *relaxed* problem. This problem deals with the averaged fields, it must have a smooth solution basically because fast oscillations of a solution of the original problem are averaged by the local problem.

The local problem is the problem of finding the extreme of the functional B upon the rectangle Ω if the materials distribution $D(x)$ as well as the fields ∇w and $\nabla\lambda$ are supposed to be periodic. It asks for an optimal displacement of materials $D(x)$ in the element of periodicity Ω (considered as a neighborhood of a point of the body $\mathcal{O}$ in the large scale). The solution of the local problem represents the value of the integrand of B on the optimal microstructures; it depends explicitly on the average fields $\langle\nabla w\rangle$, $\langle\nabla\lambda\rangle$ which are treated as known parameters. These parameters could be found numerically at each point of the optimal body by solving the relaxed optimal design problem in the large. They depend on the overall restrictions, on loading, on boundary conditions, on the shape of the domain etc.

3.2. *Statement of the problem*

Here we formulate the local problem of the optimal microstructure. Consider the periodic structure with the cell of periodicity Ω displaced in the uniform external field. For simplicity we put the volume of the cell Ω equal to 1:

$$|\Omega| = \mathrm{vol}\,(\Omega) = 1 \tag{24}$$

so the averaging over Ω is equivalent to integration over Ω:

$$\langle\cdot\rangle = \int_\Omega \cdot\,\mathrm{d}x. \tag{25}$$

We first consider the local problem of an optimal conducting composite (the elasticity problem is considered below in the same way). Let us specify, for definiteness, the set of admissible materials, namely, suppose that the cell Ω is divided into two subdomains, Ω_1 and Ω_2, and the conductivity $D(x)$ is equal

$$D(x) = \begin{cases} D_1 = d_1 I, & \text{if} \quad x \in \Omega_1, \\ D_2 = d_2 I, & \text{if} \quad x \in \Omega_2, \end{cases} \quad 0 < d_1 \leqslant d_2 < \infty \tag{26}$$

where d_1, d_2 are the scalar conductivity of components, I is the identity tensor. Also, it is convenient to assume that the volume fractions m_1, m_2 of two materials in a composite are given:

$$m_1 = \frac{|\Omega_1|}{|\Omega|}, \quad m_2 = \frac{|\Omega_2|}{|\Omega|} = 1 - m_1. \tag{27}$$

They could be determined later in order to minimize the value of the local problem. In other words, we want to find first a functional J:

$$J(\langle p\rangle, \langle q\rangle, m_1) = \min_{D(x)\in(26),(27)} \langle p\cdot D(x)\cdot q\rangle \tag{28}$$

where

$$p(x) = \nabla w, \quad q(x) = \nabla\lambda. \tag{29}$$

J depends on the averaged fields and on the volume fraction of materials in a mixture. The cost of the local problem is:

$$B(\langle p\rangle, \langle q\rangle) = \min_{m_1\in[0,1]} J(\langle p\rangle, \langle q\rangle, m_1). \tag{30}$$

We begin with the remark that the periodic in Ω fields w, λ satisfy (due to the self-adjoint nature of the operator L') the equality:

$$J(w, \lambda, D) = \int_\Omega \lambda L'(D(x))w = \int_\Omega W(p, q, D)\,\mathrm{d}x \tag{31}$$

where

$$W(p, q, D) = p \cdot D(x) \cdot q. \tag{32}$$

Similarly, we get

$$J''(w, \lambda, C) = \int_\Omega \lambda L''(D(x))w = \int_\Omega W''(u, v, C) \tag{33}$$

where the symmetric tensor fields u, v (strains) are associated with the elasticity operator L'' and are defined as

$$u(x) = \operatorname{def} w, \quad v = \operatorname{def} \lambda \quad \text{and} \quad W''(u, v, C) = u : C : v. \tag{34}$$

The differential operator def (short for deformation),

$$\operatorname{def} w = (1/2)(\nabla w + \nabla w^T), \tag{35}$$

means the symmetric part of a gradient of a vector potential (displacement) w.

The functionals J and J'' can be easily defined by introducing the tensor of effective properties of a composite. Let D_* denote the effective conductivity tensor of a mixture. It is determined by the relation

$$\langle j \rangle = \langle D(x) \cdot \nabla w \rangle = D_* \cdot \langle \nabla w \rangle, \tag{36}$$

which connects the averaged current $j = D(x) \cdot \nabla u$ and the averaged field ∇u. The functional (28) takes the form

$$J(\langle p \rangle, \langle q \rangle, m) = \langle p \rangle \cdot D_* \cdot \langle q \rangle = W_*(\langle w \rangle, \langle \lambda \rangle, D_*). \tag{37}$$

The tensor D_* is determined only by the microstructures of a composite; it belongs to the G_m-closure set mentioned above

$$D_* \in G_m U. \tag{38}$$

The cost of the local problem depends now only upon the effective tensor of a mixture, because the mean fields are supposed to be known:

$$\begin{aligned} B(\langle p \rangle, \langle q \rangle) &= \min_{m \in [0,1]} \min_{D_* \in G_m U} \langle p \rangle \cdot D_* \cdot \langle q \rangle \\ &= \min_{m \in [0,1]} \min_{D_* \in G_m U} W(\langle p \rangle, \langle q \rangle, D_*). \end{aligned} \tag{39}$$

Note, however, that we must calculate (39) directly without reference to the description of $G_m U$-set which is considered as unknown. Instead, we formulate the local problem as a variational problem of calculating the functional J.

$$J(p_0, q_0, D) = \min_{D(x) \in \mathscr{D}} \min_{p(x) \in \mathscr{P}} \max_{q(x) \in \mathscr{Q}} \langle W(p, q, D) \rangle \tag{40}$$

where the sets $\mathscr{P}$, $\mathscr{Q}$, $\mathscr{D}$ are:

$$\mathscr{P} = \{p : \nabla \times p(x) = 0, \quad \langle p(x) \rangle = p_0, \quad p(x) \text{ is } \Omega\text{-periodic}\} \tag{41}$$

$$\mathscr{Q} = \{q : \nabla \times q(x) = 0, \quad \langle q(x) \rangle = q_0, \quad q(x) \text{ is } \Omega\text{-periodic}\} \tag{42}$$

$$\mathscr{D} = \left\{ D(x) : D(x) = \begin{matrix} d_1 I, & \text{if} & x \in \Omega_1 \\ d_2 I, & \text{if} & x \in \Omega_2 \end{matrix}, \quad \frac{\operatorname{vol} \Omega_1}{\operatorname{vol}(\Omega)} = m_1, \right.$$

$$\left. \Omega = \Omega_1 \cup \Omega_2, \quad D(x) \text{ is } \Omega\text{-periodic} \right\}. \tag{43}$$

We have used here the differential restrictions $\nabla \times p(x) = \nabla \times q(x) = 0$ equivalent to the restriction of potentiality of p, q (29), because $\nabla \times p = 0$ if, and only if, p is a potential field: $p = \nabla w$. The solution depends only on local characteristics $\langle p \rangle$, $\langle q \rangle$, m_1 which are treated here as known parameters.

The local problem for the elasticity operator is formulated in the same way, with differences only in notations:

$$J''(w, \lambda, C) = \min_{C(x) \in \mathscr{C}} \min_{u \in \mathscr{U}} \max_{v \in \mathscr{V}} \langle W(u, v, C) \rangle, \tag{44}$$

where the sets $\mathscr{U}$, $\mathscr{V}$, $\mathscr{C}$ are:

$$\mathscr{U} = \{u : \operatorname{Ink} u(x) = 0, \quad \langle u(x) \rangle = u_0, \quad u(x) \text{ is } \Omega\text{-periodic}\} \tag{45}$$

$$\mathscr{V} = \{v : \operatorname{Ink} v(x) = 0, \quad \langle v(x) \rangle = v_0, \quad v(x) \text{ is } \Omega\text{-periodic}\} \tag{46}$$

$$\mathscr{C} = \left\{ C(x) : C(x) = \begin{matrix} C_1, & \text{if} & x \in \Omega_1 \\ C_2, & \text{if} & x \in \Omega_2 \end{matrix}, \quad \frac{\operatorname{vol} \Omega_1}{\operatorname{vol}(\Omega)} = m_1, \right.$$

$$\left. \Omega = \Omega_1 \cup \Omega_2, \quad C(x) - \Omega\text{-periodic} \right\} \tag{47}$$

where Ink (short for Inkompabilität) is the differential operator determined on symmetrical tensors:

$$\operatorname{Ink}(\cdot) = \operatorname{curl}(\operatorname{curl}^T(\cdot)) = \nabla \times (\nabla \times {}^T(\cdot)). \tag{48}$$

The equality (48) is equivalent to (34) because $\operatorname{Ink} u = 0$ if, and only if, $u = \operatorname{def}(\cdot)$.

The key question is to calculate analytically the quantities (40), (44); in other words, to solve explicitly the local problems which are the problems of optimization of the structure in an infinitesimal small neighborhood of a point of the designed body. According to the general concept of relaxation of the optimization problem, this analytical description is used in the solution of the global problem as the state equation of new composite medium. Note that parameters of an optimal medium are adaptive to the fields w and λ.

3.3. *An approach to the solution of the local problem*

We have mentioned already that the description of the G_m-closure set (the $G_m U$ set) gives the solution immediately. If this set is known, one could easily choose the element $D_* \in G_m U$ which provides a minimum of the functional J. However, here we discuss the straightforward way of finding the solution; the goal is to avoid unnecessary difficulties of the complete description of G_m-closure. We note that the problem of optimal rigidity of a composite has been solved (Lavrov *et al.*, 1980; Kohn and Strang, 1983; Gibiansky and Cherkaev, 1986, 1987; Allaire and Kohn, 1993) without reference to the G_m-closure problem. Instead, structures have been found which minimize the stored energy in a given mean field. In other words, a special part of the boundary of G_m-closure set has been found, which corresponds to the minimization of the form

$$\min_{D_* \in G_m U} p_0 \cdot D_* \cdot p_0, \quad \forall p_0 : \|p_0\| = 1. \tag{49}$$

Note that (49) requires the minimization of only one component of the tensor D_* (the minimal eigenvalue of the conductivity tensor, for example). By doing this minimization, we do not care what the other components of D_* are equal to. This makes the problem simpler than the problem of the G_m-closure.

Passing to the present problem, we mention that we deal with a pair of fields p and q and are interested in the minimization of the 'reaction' of a composite to the action of a pair of external fields. This means that we are interested in the description of a two-dimensional cross-section of the tensor's set $G_m U$.

Also, we mention that the state law in a medium

$$\langle j \rangle = D_* \cdot \langle p \rangle \tag{50}$$

does not change if the projection of the tensor D_* on a plane formed by vectors $\langle j \rangle$, $\langle p \rangle$ is fixed no matter what the goal functional is. These arguments show that we do not need to determine completely the G_m-closure set of effective tensors; we need only to describe the set of its two-dimensional projections.

4. OPTIMAL STRUCTURES IN THE CONDUCTIVITY PROBLEM

4.1. *New variables*

Our goal is to reduce the min–max problem (40)–(43) to the regular minimal variational problem which could be solved by the already developed methods (we mean the methods of relaxation of the non-convex minimal variational problems which have been well developed for a piece-wise quadratic integrand (see, e.g. Kohn and Strang, 1986; Milton, 1990a).

We note first that the cost functional J (44) of the local problem depends only on magnitudes of the vectors $|p_0|$ and $|q_0|$ and on the angle 2θ between them (the volume fraction m_1 is supposed to be fixed):

$$J = J(|p_0|, |q_0|, \theta) \tag{51}$$

where

$$\theta = \tfrac{1}{2}\arccos(|\langle p \rangle \cdot \langle q \rangle|). \tag{52}$$

Moreover, J is proportional to the amplitudes $|p_0|$ and $|q_0|$ of both external fields p and q and it can be rewritten in the form:

$$J(\langle p_0 \rangle, \langle q_0 \rangle, D) = |p_0||q_0| J(\langle p_* \rangle, \langle q_* \rangle, D) \tag{53}$$

where the normalized fields p_* and q_* are equal to:

$$p_* = \frac{1}{|\langle p_0 \rangle|} p, \quad q_* = \frac{1}{|\langle q_0 \rangle|} q. \tag{54}$$

Note that the functional $J(\langle p_* \rangle, \langle q_* \rangle, D)$ depends only on θ.

Now we seek a more convenient form of the functional J. We use the similarity of the fields p_* and q_*; each of them is a normalized curl-free field. Let us introduce new variables (Fig. 1):

$$a = p_* + q_*; \quad b = p_* - q_* \tag{55}$$

which obviously satisfy the differential restrictions

$$\operatorname{curl} a = \operatorname{curl} b = 0, \tag{56}$$

similar to those for p and q and can be treated as a new pair of fields. We note that due to (54) the mean values of these fields are orthogonal

$$\langle a \rangle \cdot \langle b \rangle = 0. \tag{57}$$

Indeed, we have by direct calculation

$$\langle a \rangle \cdot \langle b \rangle = (\langle p_* \rangle + \langle q_* \rangle) \cdot (\langle p_* \rangle - \langle q_* \rangle) = \langle p_* \rangle^2 - \langle q_* \rangle^2 = 0. \tag{58}$$

The moduli of the mean values of the variables $\langle a \rangle$ and $\langle b \rangle$ depend on the angle 2θ between vectors $\langle p \rangle$ and $\langle q \rangle$. The equality

$$|\langle a \rangle|^2 = (|p_*| + |q_*|)^2 = 2 + 2\cos 2\theta = 4\cos^2\theta \tag{59}$$

(and the similar equality for b) shows that the mean values $\langle a \rangle$ and $\langle b \rangle$ are equal to (see Fig. 1):

$$|\langle a \rangle| = 2\cos\theta = \sqrt{2\left(1 + \frac{p \cdot q}{|p||q|}\right)},$$

$$|\langle b \rangle| = 2\sin\theta = \sqrt{2\left(1 - \frac{p \cdot q}{|p||q|}\right)}. \tag{60}$$

The amplitudes of the fields are bounded by the relation

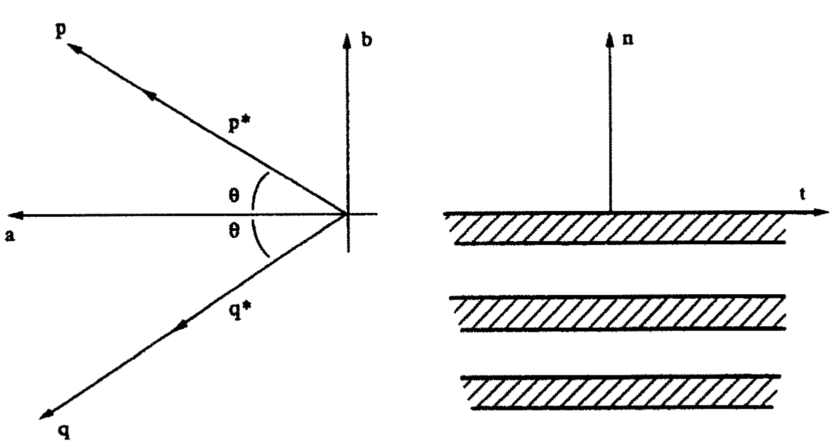

Fig. 1.

$$|\langle a\rangle|^2+|\langle b\rangle|^2 = 4. \tag{61}$$

The value $\langle W\rangle$ is transformed into the form:

$$\langle W(p,q,D)\rangle = \langle p(x)\cdot D(x)\cdot q(x)\rangle = |\langle p\rangle||\langle q\rangle|[\langle a\cdot D(x)\cdot a\rangle-\langle b\cdot D(x)\cdot b\rangle] \\ = |\langle p\rangle||\langle q\rangle|\langle W_R(a,b,d)\rangle \tag{62}$$

where

$$W_R(a,b,D) = \langle a\cdot D(x)\cdot a\rangle-\langle b\cdot D(x)\cdot b\rangle. \tag{63}$$

Note that the self-adjoint cases ($p(x) = q(x)$), see (15), or ($p(x) = -q(x)$), see (16), correspond either to $a = 0$ or $b = 0$.

The original local problem (53) is equivalent to the problem:

$$J_R(a,b,D) = \min_D \min_{a\in\mathscr{A}} \max_{b\in\mathscr{B}} W_R \tag{64}$$

where the sets $\mathscr{A}$ and $\mathscr{B}$ (see (56), (57), (60)) are:

$$\mathscr{A} = \{a:\nabla\times a = 0, \langle a\rangle = 2\cos\theta\} \tag{65}$$

$$\mathscr{B} = \{b:\nabla\times b = 0, \langle b\rangle = 2\sin\theta, \langle a\rangle\cdot\langle b\rangle = 0\}. \tag{66}$$

Indeed, the saddle point of the functional I in the variables p, q computed with a 'frozen' displacement $D(x)$ coincides with the saddle point of the functional J_R on the variables $a\in\mathscr{A}$, $b\in\mathscr{B}$ (65), (66) because the problems differ only by notations. Therefore, the values of these two problems are also equal:

$$J = J_R. \tag{67}$$

The local problems for the functional J_R could also be rewritten in terms of the effective tensors of a mixture:

$$J_R = \min_{D(x)} \min_{a(x)\in\mathscr{A}} \max_{b(x)\in\mathscr{B}} \langle a(x)\cdot D(x)\cdot a(x)-b(x)\cdot D(x)\cdot b(x)\rangle \\ = \min_{D_*\in G_m U} \langle a\rangle\cdot D_*\cdot\langle a\rangle-\langle b\rangle\cdot D_*\cdot\langle b\rangle \\ = \min_{D_*\in G_m U} W_R(\langle a\rangle,\langle b\rangle, D_*). \tag{68}$$

Now it is immediately clear that the optimal tensor D_* must be oriented so that the vectors $\langle a\rangle$ and $\langle b\rangle$ become its eigendirections; its minimal eigenvalue $d_{\min}$ must correspond to the eigenvector directed along $\langle a\rangle$ and its maximal eigenvalue $d_{\max}$ to the eigenvector directed along $\langle b\rangle$; such orientation minimizes both terms of (68). The functional J_R takes the form:

$$J_R = \min_{D_*\in G_m U} [d_{\min}\langle a\rangle^2-d_{\max}\langle b\rangle^2] = 4\min_{D_*\in G_m U} [d_{\min}\cos^2\theta-d_{\max}\sin^2\theta]. \tag{69}$$

The last formula displays the basic qualitative property of an optimal mixture: its effective tensor D_* possesses maximal difference between weighted maximal and minimal eigenvalues; in other words, these structures should be extremely anisotropic.

Coming back to the original fields $\langle p\rangle$ and $\langle q\rangle$, we see that the vectors $\langle p\rangle$ and $\langle q\rangle$ lie in the plane of maximal and minimal eigenvalues of D_* and that the direction of its minimal eigenvalue bisects the vectors $\langle p\rangle$ and $\langle q\rangle$. The functional J of the original local problem is equal to:

$$J = 2\min_{D_*\in G_mU}[d_{\min}(|\langle p\rangle||\langle q\rangle| + \langle p\rangle\cdot\langle q\rangle) - d_{\max}(|\langle p\rangle||\langle q\rangle| - \langle p\rangle\cdot\langle q\rangle)]. \quad (70)$$

4.2. *Reducing to a minimal variational problem*

Here we discuss the technique which allows us to use the regular variational methods to find optimal structures in the general case. The idea of the approach is the following: first, the min–max problem (68) is transformed to the minimal one. Then any of the known methods (for example, the translation method) could be applied to estimate the functional from below and to prove the optimality of some classes of microstructures.

To reduce the problem (68) to the minimal one, we could use the Legendre transformation on the variable b. We will replace the problem of minimization of conductivity of a composite in the direction b with the problem of minimization of its resistance in this direction. Simply speaking, we observe that the maximization of the conductance is the same as the minimization of the resistance.

So, we observe that the last term in the right hand side of the representation (68) can be replaced by:

$$E(\langle b\rangle) = \langle b\rangle\cdot D_*\cdot\langle b\rangle = \max_{\langle j\rangle} E^*(\langle j\rangle,\langle b\rangle) \quad (71)$$

where

$$E^*(\langle j\rangle,\langle b\rangle) = [2\langle j\rangle\cdot\langle b\rangle - \langle j\rangle\cdot D_*^{-1}\cdot\langle j\rangle]. \quad (72)$$

Indeed, the value $\langle j\rangle_{\text{opt}}$ which minimizes the right hand side of (71) is equal to

$$\langle j\rangle_{\text{opt}} = D_*\cdot\langle b\rangle \quad (73)$$

and the value $E^*(j_{\text{opt}})$ coincides with the left hand expression of (71). The representation (71) implies that

$$-\langle b\rangle\cdot D_*\cdot\langle b\rangle = -\max_{\langle j\rangle} E^*(\langle j\rangle,\langle b\rangle) = \min_{\langle j\rangle}\{-E^*(\langle j\rangle,\langle b\rangle)\}. \quad (74)$$

If we substitute this expression into (68), we get

$$J_R = \min_{D_*\in G_mU}[\langle a\rangle\cdot D_*\cdot\langle a\rangle - \langle b\rangle\cdot D_*\cdot\langle b\rangle = \min_{\langle j\rangle} R(\langle j\rangle)] \quad (75)$$

where $R(\langle j\rangle)$ is equal to:

$$R(\langle j\rangle) = \min_{D_*\in G_mU}[-2\langle j\rangle\cdot\langle b\rangle + \langle a\rangle\cdot D_*\cdot\langle a\rangle + \langle j\rangle\cdot D_*^{-1}\cdot\langle j\rangle]. \quad (76)$$

Now we can show that $\min_{D_*\in G_mU}R(\langle j\rangle)$ is the value of the following minimal variational problem:

$$R_*(\langle j\rangle,\langle b\rangle) = \min_{D(x)}\min_{a\in\mathscr{A}}\min_{j\in\mathscr{J}(j_0)}\int_\Omega[-2b\cdot j + a\cdot D(x)\cdot a + j\cdot D(x)^{-1}\cdot j]\,\mathrm{d}x; \quad (77)$$

where $\mathscr{J}(j_0)$ is the set of divergence-free periodic vectors with the mean value j_0:

$$\mathscr{J}(j_0) = \{j : \nabla \cdot j(x) = 0, \quad \langle j(x) \rangle = j_0, \quad j(x) \text{ is } \Omega-\text{periodic}\}. \tag{78}$$

Indeed, the first term in the right hand side of (77) can be calculated by using the compensated compactness theory (Tartar, 1978; Murat and Tartar, 1985a); this theory tells that the average of the scalar product of periodic curl-free and divergence-free vectors is equal to the scalar product of the averaged vectors:

$$\int_\Omega b(x) \cdot j(x)\,\mathrm{d}x = \langle j \rangle \cdot \langle b \rangle \tag{79}$$

the last two terms of (77) are equal to the corresponding terms of (75) due to the definition (36) of the effective tensor D_*.

Substituting the expression (77) into the functional of the local min–max problem (68), we replace it by the following minimal one:

$$J_T = \min_{\langle j_0 \rangle} \min_{D(x)} \min_{a \in \mathscr{A}} \min_{j \in \mathscr{J}(j_0)} \int_\Omega \{-2b \cdot j + a \cdot D(x) \cdot a + j \cdot D(x)^{-1} \cdot j\}\,\mathrm{d}x \tag{80}$$

or

$$J_T(\langle a \rangle, \langle b \rangle) = \min_{\langle j_0 \rangle} \min_{D(x)} \min_{a \in \mathscr{A}} \min_{j \in \mathscr{J}(j_0)} \left(-2\langle b \rangle \cdot \langle j_0 \rangle + \int_\Omega W_R\,\mathrm{d}x \right) \tag{81}$$

where

$$W_T(a, j) = a \cdot D(x) \cdot a + j \cdot D(x)^{-1} \cdot j. \tag{82}$$

Note that the variational problem for the integrand W_R is a minimal one.

The obtained form of the local problem can also be rewritten in terms of effective properties of structures:

$$\begin{aligned} J_{T_*}(j_0) &= \min_{D_* \in G_m U} \{-2\langle b \rangle \cdot \langle j \rangle + \langle a \rangle \cdot D_* \cdot \langle a \rangle + \langle j \rangle \cdot D_*^{-1} \cdot \langle j \rangle\} \\ &= -2\langle b \rangle \cdot \langle j \rangle + \min_{D_* \in G_m U} \left[d_{\min} \langle a \rangle^2 + \frac{1}{d_{\max}} \langle j \rangle^2 \right]. \end{aligned} \tag{83}$$

Clearly, the value of the last minimal variational problem is equal to the value of the original min–max problem.

We mentioned already that the last problem is well-known and that it has been well investigated and several methods have been suggested for its solution. The original problem is reduced to the problem of bounds of sum of the values of the energy of the media exposed in the external curl-free field a and in the external divergence-free field j.

The structures $D(x)$ which are optimal for the min–max functional J_R are also optimal for the minimal functional J_T and the variety of the optimal structures coincide. The exact value of the parameter j_0 may be determined only as a final step of the procedure.

Note that the minimal form of functional is symmetric in the sense that it includes the field a and the current j in the same way. One may expect this property because the original conductivity problem could be formulated in two equivalent ways by using field potential or a current potential; surely the result must be invariant of this choice.

Finally, let us show that the vectors $\langle j \rangle$ and $\langle b \rangle$ are parallel and the vectors $\langle j \rangle$ and $\langle b \rangle$ are perpendicular:

$$\langle j\rangle\cdot\langle a\rangle = 0. \tag{84}$$

Indeed, the effective tensor D_* which minimizes (83) must be oriented so that its minimal eigenvalue is directed along the vector $\langle a\rangle$ and its maximal eigenvalue along the orthogonal vector $\langle b\rangle$. This implies that b lies in eigendirection of D_* and therefore the vectors $\langle b\rangle$ and $\langle j\rangle = D_*\langle b\rangle$ are collinear; so the vectors $\langle j\rangle$ and $\langle a\rangle$ (see (57)) are orthogonal.

4.3. *Bounds*

For the problem under consideration, it is enough to use the simplest Reuss-Voigt bounds for effective tensors:

$$D_*^{-1} \geqslant \langle D(x)\rangle^{-1} = \langle d\rangle^{-1}I = \frac{1}{m_1d_1+m_2d_2}I \tag{85}$$

$$D_* \geqslant \langle D(x)^{-1}\rangle^{-1} = \langle d^{-1}\rangle^{-1}I = \frac{1}{\dfrac{m_1}{d_1}+\dfrac{m_2}{d_2}}I = \frac{d_1d_2}{m_1d_2+m_2d_1}I \tag{86}$$

which estimate the tensors D_*^{-1}, D_* as if they were independent. These estimates came from the algebraic inequalities valid for all vectors a and b:

$$\langle a\rangle\cdot D_*\cdot\langle a\rangle \geqslant \langle a\rangle\cdot\frac{d_1d_2}{m_1d_2+m_2d_1}\cdot\langle a\rangle = \langle d^{-1}\rangle^{-1}\langle a\rangle^2, \tag{87}$$

$$\langle j\rangle\cdot D_*^{-1}\cdot\langle j\rangle \geqslant \langle j\rangle\cdot\frac{1}{m_1d_1+m_2d_2}\cdot\langle j\rangle = \langle d\rangle^{-1}\langle j\rangle^2. \tag{88}$$

To find the lower estimate the functional $J_T'^*$ let us substitute (87) and (88) into (83). We end up with the inequality:

$$J_T^* \geqslant \langle -a\cdot j\rangle + \langle a\rangle\cdot\langle D^{-1}(x)\rangle\cdot\langle a\rangle + \langle j\rangle\cdot\langle D(x)\rangle^{-1}\cdot\langle j\rangle = -\langle b\rangle\cdot\langle j\rangle + \frac{d_1d_2}{m_1d_2+m_2d_1}\langle|a|\rangle^2 + \frac{1}{m_1d_1+m_2d_2}\langle|j|\rangle^2. \tag{89}$$

The inequality (89) is valid for all composites independent of their structure. This bound is also exact; we will show that the appropriate oriented layered composites can be used to provide the minimal value of functional J_T^*.

4.4. *Optimal structures*

Let us consider a laminate composite assembled from two materials with scalar conductivity d_1 and d_2, let $\mathbf{n}$, $\mathbf{t}$ be a normal and a tangent (or tangents) to layers. It is known (see, e.g. Lurie and Cherkaev, 1986) that the effective conductivity of such a structure is described by an anisotropic tensor D_{lam}:

$$D_{\text{lam}} = \langle d\rangle\mathbf{t}\otimes\mathbf{t} + \langle d^{-1}\rangle^{-1}\mathbf{n}\otimes\mathbf{n} = (m_1d_1+m_2d_2)\mathbf{t}\otimes\mathbf{t} + \left(\frac{m_1}{d_1}+\frac{m_2}{d_2}\right)\mathbf{n}\otimes\mathbf{n} \tag{90}$$

where $\otimes$ means a tensor product (for two arbitrary vectors $a = [a_1, a_2, \ldots, a_n]$ and $b = [b_1, b_2, \ldots, b_m]$ we have: $a\otimes b = c$, where c is a $n\times m$ matrix of the type

$$c_{ij} = a_ib_j). \tag{91}$$

The tensor of effective moduli of a laminate D_{lam} has one minimal eigenvalue equal to

$$\frac{d_1 d_2}{m_1 d_2 + m_2 d_1} \tag{92}$$

and the others eigenvalues equal to

$$m_1 d_1 + m_2 d_2. \tag{93}$$

The inverse tensor D_{lam}^{-1} has the minimal eigenvalue equal to

$$\frac{1}{m_1 d_1 + m_2 d_2}. \tag{94}$$

If laminates are oriented in such a way that the normal **n** coincides with the direction of the field $\langle b \rangle$, and the tangent **t** with the direction of $\langle a \rangle$, then they obviously represent the optimal structure of a composite. Indeed, the value of J_T^* coincides with the estimates, if it is calculated for a laminate composite, which is oriented in a way described above. Thus the solution of the local problem is found.

Remark 2. We could solve the min–max problem (68) directly without passing to the minimal one in the following way. Due to orthogonality of the fields $\langle a \rangle$ and $\langle b \rangle$ it is clear that the minimum value of functional corresponds to such orientation of the principle axes of the tensor D_* that they coincide with the directions of vectors $\langle a \rangle$ and $\langle b \rangle$; the maximal eigenvalue must be oriented along the vector $\langle b \rangle$, and the minimal eigenvalue must be oriented along the vector $\langle a \rangle$. Now it is clear immediately that for the conductivity problem only layered composites are the appropriate ones, because only they possess simultaneously the maximal conductance in some direction(s) (along the layers) and the minimal conductance in the perpendicular direction (across the layers).

4.5. *The relaxed problem*

Let us finally calculate the optimal value of the volume fractions of materials in the laminates and the cost of the functional B_R. We have $(m_2 = 1 - m_1)$:

$$B_R = \min_{m_1 \in [0,1]} d_{\min}(m_1) a^2 - d_{\max}(m_1) b^2 = \min_{m_1 \in [0,1]} \frac{d_1 d_2}{m_1 d_2 + (1 - m_1) d_1} a^2 - (m_1 d_1 + (1 - m_1) d_2) b^2. \tag{95}$$

It is easy to show by direct calculation of the derivative (dB/dm_1) that the optimal value m_1^{opt} of m_1 is equal to

$$m_1^{\text{opt}} = \begin{cases} \emptyset & \text{if} \quad b/a = \cot\theta \leqslant \sqrt{\dfrac{d_1}{d_2}} \\ \dfrac{\sqrt{d_1}\cot\theta}{\sqrt{d_1} + \sqrt{d_2}} & \text{if} \quad \sqrt{\dfrac{d_1}{d_2}} \leqslant b/a = \cot\theta \leqslant \sqrt{\dfrac{d_2}{d_1}} \\ 1 & \text{if} \quad b/a = \cot\theta \geqslant \sqrt{\dfrac{d_2}{d_1}} \end{cases} \tag{96}$$

and the value of functional is equal to

$$B_R = \begin{cases} d_2(a^2-b^2) & \text{if} \quad b/a = \cot\theta \leqslant \sqrt{\dfrac{d_1}{d_2}} \\ \sqrt{d_1 d_2}\sin 2\theta - (d_1-d_2)(1-\cos 2\theta)/2 & \text{if} \quad \sqrt{\dfrac{d_1}{d_2}} \leqslant b/a = \cot\theta \leqslant \sqrt{\dfrac{d_2}{d_1}} \\ d_1(a^2-b^2) & \text{if} \quad b/a = \cot\theta \geqslant \sqrt{\dfrac{d_2}{d_1}}. \end{cases} \tag{97}$$

Passing to the original notations we get the value of the local problem in the form:

$$J = \begin{cases} d_1\nabla w\cdot\nabla\lambda & \text{if} \quad \cot\theta \leqslant \sqrt{\dfrac{d_1}{d_2}} \\ \sqrt{d_1 d_2}|\nabla\lambda\times\nabla w| - (d_1-d_2)(|\nabla\lambda||\nabla w| - \nabla\lambda\cdot\nabla w)/2 & \text{if} \quad \sqrt{\dfrac{d_1}{d_2}} \leqslant \cot\theta \leqslant \sqrt{\dfrac{d_2}{d_1}} \\ d_2\nabla w\cdot\nabla\lambda & \text{if} \quad \cot\theta \geqslant \sqrt{\dfrac{d_2}{d_1}}. \end{cases} \tag{98}$$

This shows that the optimal concentration of materials in the laminations depends on the angle θ: if the vectors ∇w and $\nabla\lambda$ are nearly parallel, then the good conductor is used as optimal, if they are close to antiparallel, then the bad conductor is the best, and if these vectors are close to orthogonal, then the optimal structures are laminates.

To solve the problem completely, it remains to substitute the value of the local problem into the initial functional (9) and find the Euler equations of the global problem

$$I_A = \min_w \max_\lambda \int_{\mathcal{O}} [wr + \lambda q + B(w,\lambda)]\,dx. \tag{99}$$

The Euler equations of it are:

$$\delta_\lambda J = q - \nabla\cdot\frac{\partial B}{\partial\nabla\lambda} = 0; \tag{100}$$

$$\delta_w J = r - \nabla\cdot\frac{\partial B}{\partial\nabla w} = 0. \tag{101}$$

The last equations depend only on w and λ; they should be solved numerically to complete the solution of the problem.

Remark 3. It should be mentioned that the solution of the present problem was obtained in different ways in a number of papers. Lurie (1993) has investigated the asymptotic case ($d_0 = 0$) and has found that an eigendirection of the effective tensor must coincide with the bisector of the fields p and q, i.e. with the vector b; Tartar (Tartar, 1978; Raitum, 1983) got the same result using an approach close to the G-closure approach; Gibiansky *et al.* (1988) have found the solution to the problem using the necessary condition of optimality of orientation of laminates and the description of the G-closure; recently Lurie (1990) has solved the problem by building a quasi-saddle estimate of the functional. However, we believe that the approach of reducing the problem to the minimal form presented here is more than exercise because, due to its universality, it allows the regular consideration of the problem of structural design in more complicated cases and because it provides qualitative information about the class of optimal structures. We illustrate this in the next section by an example of optimal design of elastic structures.

Remark 4. It is clear that the obtained results could be easily extended to the more than two component mixtures, which are considered here for simplicity. The result in the general case is the same; optimal structures are but laminates, which bisect the directions of the fields $\langle p\rangle$ and $\langle q\rangle$.

5. OPTIMAL STRUCTURES IN PLANAR ELASTICITY

5.1. *New variables and the minimal variational problem*

Clearly, the outlined procedure can be applied for the problems of elasticity as well. We restrict ourselves here to the problem of planar elasticity. Again, we consider a local min–max problem

$$\min_C \min_u \max_v \int_\Omega u : C : v \, dx \tag{102}$$

where u, v are the symmetric normalized two by two tensors of strains

$$\|u\| = \|v\| = 1. \tag{103}$$

The norm of a tensor is determined as

$$\|a\| = \sqrt{\sum_{i,j} a_{ij} a_{ji}} = \sqrt{a : a}. \tag{104}$$

As before, we transform (102) by introducing new variables,

$$\varepsilon = u + v, \quad \varepsilon' = u - v. \tag{105}$$

The new form of the problem is (compare with (47))

$$\min_C \min_{\varepsilon \in \mathscr{A}'} \max_{\varepsilon' \in \mathscr{B}'} \int_\Omega \varepsilon : C : \varepsilon - \varepsilon' : C : \varepsilon' \tag{106}$$

where the sets $\mathscr{A}'$, $\mathscr{B}'$ are

$$\mathscr{A}' = \{\varepsilon : \operatorname{Ink} \varepsilon = 0, \quad \langle \varepsilon \rangle = u_0 + v_0, \quad \varepsilon(x) \text{ is } \Omega\text{-periodic}\} \tag{107}$$

$$\mathscr{B}' = \{\varepsilon : \operatorname{Ink} \varepsilon'(x) = 0, \quad \langle \varepsilon'(x) \rangle = u_0 - v_0, \quad \varepsilon'(x) \text{ is } \Omega\text{-periodic}, \quad \langle \varepsilon \rangle : \langle \varepsilon' \rangle = 0\}, \tag{108}$$

and the set $\mathscr{C}$ was determined earlier (47). Again the problem is to find such a structure of composite which minimizes the difference of weighted energy density caused by two orthogonal strain fields ε and ε'.

The transformation of this problem to the minimal one leads to the following problem: find the microstructure of a composite, assembled from two given materials, which minimizes the sum of stored energy, generated by some uniform strain ε and the complementary energy, generated by an orthogonal uniform stress $\sigma = C : \varepsilon'$:

$$J^{el} = \min_{\varepsilon \in \mathscr{A}} \min_{\sigma \in \Sigma} \min_{C \in \mathscr{C}} \left[-2\langle \sigma \rangle : \langle \varepsilon' \rangle + \int_\Omega W^{el}(\varepsilon, \sigma, C) \, dx \right] \tag{109}$$

where

$$W^{el}(\varepsilon, \sigma, C) = \varepsilon : C(x)^{-1} : \varepsilon + \sigma : C(x) : \sigma \tag{110}$$

$$\Sigma = \{\sigma : \sigma = \sigma^T, \quad \operatorname{div} \sigma = 0, \quad \langle \sigma \rangle = \sigma_0, \quad \sigma_0 : \varepsilon_0 = 0, \tag{111}$$

and the sets $\mathscr{A}$, $\mathscr{C}$ are determined above. In other words, the optimal microstructure must minimize the sum of a rigidity under one loading and a compliance under another orthogonal loading.

This time the Reiss–Voigt inequalities provide only estimates of the sum of the values of stored energy (but not fine bounds of this sum). The more accurate methods could be applied here like the translation estimates or the Hashin–Shtrikman type estimates. These estimates have been used by Gibiansky and Cherkaev (1986) and Allaire and Kohn (1993) to get the exact value for each term of (110), i.e. to get the exact separate estimates for the densities of the energy and the complementary energy. The estimates depend of the ratio of the bulk and shear parts of the external loadings ε and σ. As opposed to the Reiss–Voigt estimates, the translation method also provides the coupled estimates of the maximal rigidity and the maximal compliance in two orthogonal fields, as was demonstrated in the recent paper (Cherkaev and Gibiansky, 1993).

Remark 5. The translation method has been successively applied to various problems of elastic composites; by using it, plane and three-dimensional elastic structures with maximal rigidity and with maximal compliance have been found, as well as the structures which correspond to the bounds in the rigidity of an elastic polycrystal, etc. The translation estimates for minimizing a sum of elastic energies and of complementary energies (109) were obtained in Cherkaev and Gibiansky (1993). Although the final calculations have been made under the assumption of the isotropy of the effective tensor, which is surely not the case for the considered problem, the procedure of estimating can be applied to it as well.

However, in this paper we restrict ourself to particular situations which allow us to bound the energy in a simple way and which do not need the use of the general technique of translation estimates. Our goal is to show that, at least in these cases, the optimal structures exist and that they can be effectively found.

5.2. *Examples of optimal elastic structures*

Similarly to the conductivity problem, we find immediately the optimal structures in several particular cases of the elastic optimal design problem. By doing these examples, we are going to show several classes of optimal structures.

Note first, that the optimal microstructure should be determined by parameters of a pair of orthonormalized tensors ε and ε' and by the volume fraction m_1. On the other hand, any pair of orthonormalized tensors σ and σ' are determined by three scalar characteristics, and an orientation of the axis (the last parameter is of no importance for us). The type of optimal microstructure should be determined by these parameters and by the volume fractions, and its orientation in the labor axes is determined by the orientation of the fields. The complete description of the class of optimal structures has not been found yet, but we can demonstrate several subclasses of optimal structures which correspond to some specific families of pairs ε and ε'.

Firstly, we know already the structures which minimize the stored energy in the field ε or in the field σ, which means that we know the solution of (109) in two limiting cases,

$$u = v \quad \text{or} \quad \varepsilon = 0 \tag{112}$$

(the first self-adjoint problem) and

$$u = -v \quad \text{or} \quad \varepsilon' = 0 \tag{113}$$

(the second self-adjoint problem). We assume here that materials can be ordered as following

$$C_1 \leqslant C_2. \tag{114}$$

The known structures of elastic composites which minimize the stored energy in the field σ are the so-called "matrix laminates of second-rank" (Gibiansky and Cherkaev 1986). The structures optimal for the problem (112) are obtained as laminates (with normal n_1) assembled from the first material C_1 and homogenized laminates with normal n_2 made from first C_1 and second C_2 material (see Fig. 2). We see that the material C_2 is situated in inclusions, and the material C_1 forms an envelope around these inclusions. The relative concentrations of the first material in laminates of the first- and second-rank v_1, v_2 can be changed assuming that the total volume fractions of materials should be kept constant.

The set of structures which gives the solution in the second case (113) is also known. This time the problem is reduced to the problem of minimization of the complementary energy which was solved in Gibiansky and Cherkaev (1986); the optimal microstructures are again the second-rank matrix laminates, but now the material C_2 forms the envelope and the material C_1 is situated in inclusions.

The effective compliance tensor C^{2r} of second-rank laminates (the material C_1 forms the envelope) has the form (Francfort and Murat, 1991; Gibiansky and Cherkaev, 1987)

$$C^{2r} = C_1 + m_2\left[(C_2 - C_1)^{-1} + c\sum_{i=1}^{2} v_i n_1 \otimes n_i \otimes n_i \otimes n_i\right]^{-1} \tag{115}$$

where C_1 and C_2 are fourth-order tensors of compliances of two initially given isotropic materials. The constant c is equal to

$$c = \frac{m_1}{m_2(n_i \otimes n_i : C_1 : n_i \otimes n_i)} = \text{const}\ \forall i; \tag{116}$$

it is independent of the direction of the unit vector n_i because of isotropy of the tensor C_1. The parameters v_i represent the normalized concentrations of material C_2 in laminates in the first- and second-rank of the structure:

$$v_1 + v_2 = 1, \quad v_1 \geqslant 0, \quad v_2 \geqslant 0. \tag{117}$$

It has been proved by Gibiansky and Cherkaev (1986) (see also Arraire and Kohn, 1993)

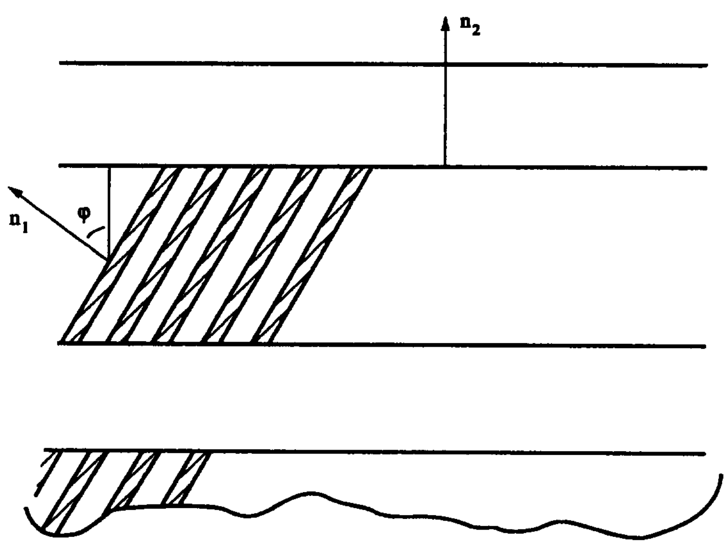

Fig. 2.

that second-rank laminates correspond to the optimal rigidity if the three following conditions hold:

(i) the vectors n_1 and n_2 are orthogonal:

$$n_1 \cdot n_2 = 0, \tag{118}$$

(ii) they are oriented along principle axis of the tensor σ,

(iii) the relative concentrations v_i are chosen in a proper way, depending on the ratio of eigenvalues of σ.

Let us demonstrate now that these classes of structures give a solution of a structural optimization problem for larger classes of values of parameters of local problems than the self-adjoint cases (112), (113).

Note that second-rank laminates (115) possess maximum compliance in the trace free tensor "direction" $\mathbf{z}$:

$$\mathbf{z} = \frac{1}{\sqrt{2}}(n_1 \otimes n_2 + n_2 \otimes n_1), \quad \|\mathbf{z}\| = 1. \tag{119}$$

This compliance is equal to:

$$\mathbf{z} : C^{2r} : \mathbf{z} = \mathbf{z} : (m_1 C_2 + m_2 C_2) : \mathbf{z} \tag{120}$$

no matter what the values of parameters v_i are. This means that matrix laminates store a minimum of complementary energy in a trace free strain field ε. At the same time, as it has been mentioned above, they store the minimal elastic energy in an arbitrary field σ if the parameters v_i are chosen in an appropriate way and if σ is orthogonal to ε. Therefore, they provide a solution of the general problem (109) if the field ε is trace free but the field σ is arbitrary.

Example. As an example let us determine the optimal structure of a thin circular cylindrical shell loaded at its edges by a uniform loading directed along the cylindrical axes. Suppose that we want to maximize the deflection of the cylinder in the radial direction; in other words, we want to maximize the increase of the radius of the shell.

Due to symmetry of the problem, we look for a uniform structure which does not depend on the coordinates of the surface. In the microstructure scale we can also neglect the curvature of the surface. Then we find the composite which maximizes its displacement in the (circumferential) direction orthogonal to the one-axis loading. Physically, we could say that such a structure has maximum possible value of the Poisson ratio (the last statement is not precisely correct because the optimal medium is anisotropic).

We suppose that the structure is placed into uniform strain field a equal to

$$a = \mathbf{i} \otimes \mathbf{i} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{121}$$

and the minimizing strain is

$$b = -\mathbf{j} \otimes \mathbf{j} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}. \tag{122}$$

(The last expression actually means that the $\mathbf{j} \otimes \mathbf{j}$ component of strain is maximized.) We then have

$$\varepsilon = \mathbf{i}\otimes\mathbf{i}-\mathbf{j}\otimes\mathbf{j} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad \varepsilon' = \mathbf{i}\otimes\mathbf{i}+\mathbf{j}\otimes\mathbf{j} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{123}$$

It is clear that $\mathrm{Tr}\,\varepsilon = 0$, therefore the optimal composite is a second-rank matrix laminate. Let us find its parameters. The structure that minimizes the term $\sigma : C : \sigma$ is a cubic symmetric second-rank laminate with parameters $v_1 = v_2 = \frac{1}{2}$ (see Gibiansky and Cherkaev, 1986). The effective tensor of that structure transforms the bulk field ε' into the bulk field σ:

$$\sigma = C^{-1} : \varepsilon' = \alpha(\mathbf{i}\otimes\mathbf{i}+\mathbf{j}\otimes\mathbf{j}) \tag{124}$$

where α is a proportionality coefficient. The value of the term $\sigma : C : \sigma$ in the functional (110) depends on the bulk modulus only and, therefore, it is independent of orientation of the structure. The orientation is to be determined by the equality (119), (120) which shows that normal n_1, n_2 to layers in the structure must be oriented along the bisectors $\mathbf{i}+\mathbf{j}$, $\mathbf{i}-\mathbf{j}$ of the axes $\mathbf{i}$, $\mathbf{j}$.

So we have found that the optimal structure is a second-rank laminate turned by the angle 45° to the cylindrical axes. It can be imitated by two families of orthogonal helices which reinforce the envelope.

Other types of optimal structures are simple laminates. Indeed, they show minimum compliance in the uniaxial stress field along the layers and minimum rigidity in all orthogonal tensor directions. Therefore, laminates are optimal if the field σ is uniaxial and the field ε is arbitrary. For example, laminates assembled from materials with zero Poisson ratio provide maximum shear strain under hydrostatic stress. Indeed, we suppose that the structure is placed into uniform strain field a' equal to

$$a' = \mathbf{i}\otimes\mathbf{i}+\mathbf{j}\otimes\mathbf{j} \tag{125}$$

and the minimizing strain is

$$b' = \mathbf{i}\otimes\mathbf{i}-\mathbf{j}\otimes\mathbf{j}. \tag{126}$$

(This means that the strain component along $-b'$ is maximized.) We then have

$$\varepsilon = \mathbf{i}\otimes\mathbf{i}, \quad \varepsilon' = \mathbf{j}\otimes\mathbf{j}. \tag{127}$$

The best structure is a laminate oriented along the $\mathbf{j}$ axis. Indeed, the stress field $\sigma = C^{-1}\varepsilon'$ is uniaxial and the compliance is minimal along this direction.

It is easy to use physical arguments to see why this structure is optimal. Indeed, a square piece of laminated material, being extremely anisotropic, deforms under hydrostatic pressure into a rectangle with maximal ratio between its sides.

5.3. *Asymptotic optimal structures*

Here we are going to demonstrate a class of composite structures which solve another special case of the problem. This time we pass to an asymptotic: we suppose that microstructures are assembled from two materials; one of them possesses infinitely small rigidity, and the other possesses infinitely large rigidity

$$\|C_1\| \to 0, \quad \|C_2\| \to \infty. \tag{128}$$

The estimates (110) became a trivial form:

$$I = \int_\Omega [\sigma : C_*(x) : \sigma + \varepsilon : C_*^{-1}(x) : \varepsilon]\,\mathrm{d}x \geqslant 0. \tag{129}$$

Again we are not going to use the translation estimates directly; instead we show structures with the effective tensor C_{opt} which possess zero rigidity by applying any stress field σ, and zero compliance (or infinite rigidity) by applying any other strain field ε orthogonal to σ. For these structures, the functional (129) vanishes:

$$\sigma : C_{\text{opt}} : \sigma = 0, \quad \varepsilon : C_{\text{opt}}(x) : \varepsilon = 0. \tag{130}$$

The microstructures with the required properties have been found in the paper by Cherkaev and Milton (1993) (following the earlier approach by Milton, 1993) where an even more general problem was solved: the variety of asymptotic microstructures has been described which corresponds to any positively defined effective tensor of compliance.

Note that the volume fraction of materials in a mixture is of no importance in the asymptotic case and the problem of optimal structures becomes a purely geometrical one. Physically, this follows from the fact that the infinitely thin element of structure made from absolutely rigid or absolutely soft material may make the structure's effective compliance infinitely large or infinitely small, no matter what the volume fraction of this element is.

Let us examine first the matrix second-rank composites with arbitrary orientation of the normal n_1 and n_2 (Fig. 2), and let us define the angle between n_1 and n_2 as ϕ

$$\cos\phi = n_1 \cdot n_2. \tag{131}$$

The effective tensor properties of such structures (115) become, under assumptions, (128) an asymptotic form:

$$C^{-1} = \frac{c}{m_2}(n_1 \otimes n_1 \otimes n_1 \otimes n_1 + n_2 \otimes n_2 \otimes n_2 \otimes n_2) \tag{132}$$

where (see (116))

$$c = \frac{m_1}{(n \otimes n : C_2 : n \otimes n)}, \quad (c \to \infty). \tag{133}$$

The expression on the right hand side is a singular tensor with two infinite eigenvalues and with one zero eigenvalue. The last one corresponds to the eigentensor $\mathbf{z}$, orthogonal to both tensors

$$a_1 = n_1 \otimes n_1, \quad a_2 = n_2 \otimes n_2. \tag{134}$$

We find $\mathbf{z}$ from the relations:

$$\mathbf{z} : a_1 = \mathbf{z} : a_2 = 0. \tag{135}$$

Let us direct the principle axis n, t, of $\mathbf{z}$ along the bisectrices of the normal and tangential to layers:

$$n = \frac{1}{\|n_1 + n_2\|}(n_1 + n_2), \quad t = \frac{1}{\|n_1 - n_2\|}(n_1 - n_2). \tag{136}$$

In that basis $\mathbf{z}$ has the form:

$$\mathbf{z} = \begin{pmatrix} \cos\psi & 0 \\ 0 & \sin\psi \end{pmatrix}. \tag{137}$$

The scalar products (135) are equal to

$$\mathbf{z}:a_1 = \mathbf{z}:a_2 = \cos^2\phi/2\cos\psi + \sin^2\phi/2\sin\psi. \tag{138}$$

The parameter ψ depends on the angle ϕ between normals of layers of first- and second-rank; it must be chosen to cause both expressions above to vanish:

$$\tan\psi = -\tan^2\phi. \tag{139}$$

If the angle ϕ is varied in the interval $[0, \pi/2]$ then the variety of tensors $\mathbf{z}$ form a set of normalized symmetric tensors; the tensors are arbitrary but with non-positive determinant:

$$\det\mathbf{z} = \sin\psi\cos\psi = -\tan^2\phi\sin^2\psi \leqslant 0. \tag{140}$$

The effective compliance tensor $S = C^{-1}$ of such class of structures is inverse to the expression in the right hand side of (132); it has one infinite eigenvalue (which corresponded to the eigentensor $\mathbf{z}$) and two eigenvalues equal to zero,

$$S = C^{-1} = \infty\mathbf{z}\otimes\mathbf{z}. \tag{141}$$

In other words, the matrix second-rank laminate structure has zero compliance in the directions $n_1\otimes n_1$ and $n_2\otimes n_2$ and in any direction of their convex combinations $a = \lambda n_1\otimes n_1 + (1-\lambda)n_2\otimes n_2$, and has infinite compliance (zero resistance) in a direction $\mathbf{z}$ orthogonal to all tensors a simultaneously. This structure allows the only mode of strain ε proportional to $\mathbf{z}$:

$$\varepsilon = \kappa\mathbf{z} \tag{142}$$

where κ is an arbitrary constant.

Clearly a tensor of this class corresponds to zero value of the functional (109) if it is oriented in such a way that the tensor $\mathbf{z}$ is proportional to the field ε, the orthogonal field σ obviously corresponds to infinite rigidity. Note again that the tensor $\mathbf{z}$ is not arbitrary because its determinant should be negative. Therefore, the described structures provide solutions for optimal problems only for the points where the tensor ε has a non-positive determinant.

To find structures which complete the set of optimal composites, it is enough to refer to the construction suggested by Milton (1993). Following his approach, let us consider a "herring-bone structure"; a laminate structure assembled from differently oriented anisotropic materials which we treat as given materials. The concentrations of these anisotropic materials in the structure were denoted by μ and $1-\mu$.

Let us use the described second-rank matrix laminates as given materials for the herring-bone structure (Fig. 3). Thus, each compound can possess only one mode of strain $\varepsilon = \kappa\mathbf{z}$, where κ is some real number, and the tensor $\mathbf{z}$ (such that $\det\mathbf{z} \leqslant 0$) is determined by the structure of second-rank laminates. A herring-bone structure possesses strains ε which are convex combinations

$$\varepsilon = \mu\varepsilon_1 + (1-\mu)\varepsilon_2, \quad 0 \leqslant \mu \leqslant 1 \tag{143}$$

of strains ε_1, ε_2 in the first and second layers, assuming that these strains are compatible, that is that their tangential components are equal:

$$t\cdot[\varepsilon_1 - \varepsilon_2]\cdot t = 0; \tag{144}$$

where t is the tangent to the layers.

Let us now choose the materials in layers as a second-rank laminate structure which differ from one another only by reflection in the tangential t; the strains allowed in these materials have in a basis n, t the following form (see (142)):

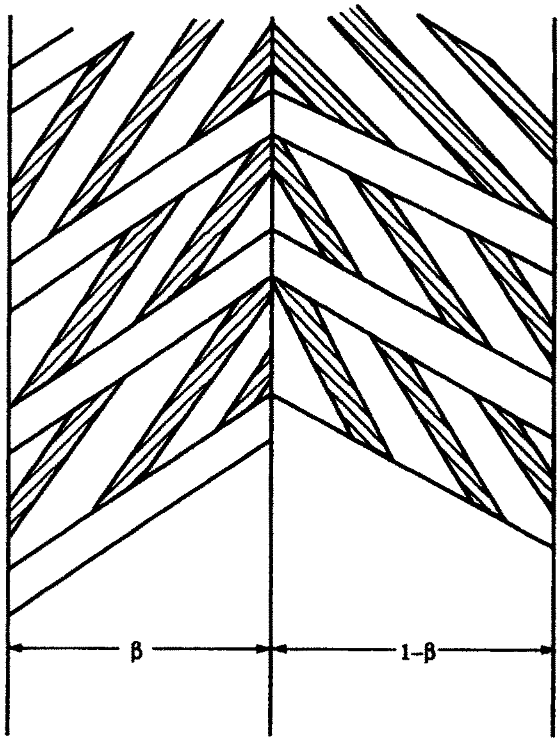

Fig. 3.

$$\varepsilon_1 = \kappa_1\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}, \quad \varepsilon_2 = \kappa_2\begin{pmatrix} \alpha & -\beta \\ -\beta_2 & \gamma \end{pmatrix} \tag{145}$$

where κ_1, κ_2 are arbitrary constants, α, β, γ are the normalized parameters of the strain in the second-rank matrix composite:

$$\alpha\beta - \gamma^2 \leqslant 0, \quad \alpha^2 + 2\beta^2 + \gamma^2 = 1. \tag{146}$$

The equality of compatibility (144) implies

$$\kappa_1 = \kappa_2 = \kappa. \tag{147}$$

The strain (143) which corresponds to the infinite compliance of the structure is founded by substitution of (147), (145) into (143); it has a form

$$\varepsilon = \kappa\begin{pmatrix} \alpha & (1-2\mu)\beta \\ (1-2\mu)\beta & \gamma \end{pmatrix}; \tag{148}$$

all other orthogonal strains are equal to zero. Therefore, the effective tensor of a herring-bone structure has the form (141) where

$$\mathbf{z} = \frac{1}{\|\varepsilon\|}\varepsilon. \tag{149}$$

It has one infinite eigenvalue (corresponding to the described eigentensor **z**) and two zero eigenvalues.

It is easy to show, following Milton (1993), that the variety of eigentensors **z** formed by varying the parameters of a herring-bone structure includes all possible normalized symmetric tensors, both with positive and negative determinants. We may, for example, vary only the concentration μ of the layers and fix somehow the other parameters α, β, γ of structure. Let us fix the orientation of the second-rank laminates in such a way that the direction of the normal in the herring-bone structure bisects the eigendirections of the tensors ε_1 and ε_2. This leads to the representation:

$$\alpha = \gamma = 1/2\,\mathrm{Tr}\,\varepsilon_1 = 1/2\,\mathrm{Tr}\,\varepsilon_2. \tag{150}$$

(We assume also that $\mathrm{Tr}\,\varepsilon_1 \neq 0$, which means that the layers in the second-rank laminates are not orthogonal.) Then the variety of the averaged normalized tensors $\mathbf{z}(\mu)$ (see (149))

$$\mathbf{z}(\mu) = \varepsilon \frac{1}{\|\varepsilon\|} = \frac{1}{\sqrt{\alpha^2 + [(1-2\mu)\beta]^2}} \begin{pmatrix} \alpha & (1-2\mu)\beta \\ (1-2\mu)\beta & \alpha \end{pmatrix}, \tag{151}$$

includes all positively defined tensors when μ changes its value from zero to 1/2. Indeed, the determinant of $\mathbf{z}(\mu)$ is equal to

$$\det \mathbf{z}(\mu) = \frac{\alpha^2 - [(1-2\mu)\beta]^2}{\alpha^2 + [(1-2\mu)\beta]^2}; \tag{152}$$

it varies from one to zero, when

$$\mu \in [1/2, \quad (1-(\alpha/\beta)^2)/2]. \tag{153}$$

The described structures provide the solution of our problem: they do not resist an arbitrary strain ε because they can be oriented in such a way to make the compliance against that strain infinite; the orthogonal directions correspond to zero compliance because no strain arises in that direction. This means that the herring-bone structures provide an example of the class of composite media which have zero compliance in an arbitrary tensor direction and infinite compliance in any orthogonal one.

One can be sure that this class contains an element which solves the problem (109) because the sum of the stored energy and complementary energy in a properly oriented structure of this kind is zero.

6. DISCUSSION

6.1. *The relaxation method*

We have outlined the basic steps of the described method of relaxation of the optimal design problem. We assume that a material has a linear low state (like Ohm's and Hook's laws). The problem is described by elliptic equations; the shape of the domain, boundary conditions and external loadings (right hand side terms) are fixed, the minimizing functional is lower weakly semi-continuous. The problem asks for optimal distribution of several given materials in the body. To solve this problem one can follow the scheme:

(i) Formulate the local problem as a variational min–min–max problem for the integral of the bilinear form of the field and of the Lagrange multiplier; the coefficients of the form represent properties of materials.

(ii) Normalize the bilinear form and transform it to the diagonal form by introducing new potentials. End up with min–min–max problem for the integral of difference of two quadratic forms (or for the difference of two energies).

(iii) Use Legendre transformation to reduce the problem to the minimal variational problem, end up with the problem of minimization of the sum of the energy and the complementary energy upon all structures.

(iv) Use the translation method or similar variational methods to find the value of the minimized functional (or its lower estimate) and use lamination technique to build a minimizing sequence (i.e. the optimal microstructure).

(v) Return to the original notations, formulate and numerically solve a relaxed variational problem "in the large" to find the actual distribution of optimal microstructures throughout the body and the optimal distribution of volume fractions of the materials in microstructures.

6.2. Link with the G_m*-closure problem*

The described procedure does not require the description of the G_m-closure. We need only the description of special class of structures—those which are only candidates for use in the optimal design.

Let us compare these two problems. The G_m-closure problem requires us to estimate the sum of three (plane elasticity) or six (three-dimensional elasticity) values of the energy and/or complementary energy to establish each component of the G_m-closure set. On the contrary, here we estimate the sum of one energy and one complementary energy and we do not care about the properties of the composite in the directions orthogonal to both fields. This observation makes the procedure much easier, as it was easier to establish the optimal structure for the self-adjoint optimization problem, where only the properties in one direction were important.

Informally speaking, difficulties grow exponentially with the number of estimating terms. Therefore, the suggested approach seems more reasonable to obtain results in a comparatively short time and in explicit analytical form. Simultaneously, we will get the description of some "ribs" of the set of invariants of the G_m-closure set of effective tensors, which is useful for the complete description of this set.

6.3. Link with the problems of optimal dissipative media

The non-self-adjoint optimization problem for a self-adjoint operator $L(D)$ could be linked with the simplest problem of maximization of the energy of a dissipative medium. We could consider both fields w and λ as the real and imaginary parts of some complex potential v

$$\psi = w + \mathbf{i}\lambda \tag{154}$$

and the functional of the local problem B, as the real part of an energy stored in the media with complex-valued properties tensor $0+\mathbf{i}D$. Note that the imaginary part of complex-valued properties tensor describes the rate of dissipation of the energy of the dissipative medium by periodic loading. For example, the non-self-adjoint optimization problem for the conducting medium (7) is equivalent to the problem of minimizing the rate of dissipation of the energy of an absorbing medium:

$$J = \int_{\mathcal{O}} [\nabla\psi\cdot(\mathbf{i}D)\cdot\nabla\psi] = \int_{\mathcal{O}} [-2\nabla w\cdot D\cdot\nabla\lambda + \mathbf{i}(\nabla w\cdot D\cdot\nabla w - \nabla\lambda\cdot D\cdot\nabla\lambda)]. \tag{155}$$

Similarly, the optimization problem for an elastic medium is equivalent to the problem of minimizing the rate of dissipation of the energy of a viscous medium:

$$J'' = \int_{\mathcal{O}} [\operatorname{def} v : (\mathbf{i}C) : \operatorname{def} v] = \int_{\mathcal{O}} \operatorname{def} w : C : \operatorname{def}\lambda + \mathbf{i}(\operatorname{def} w : C : \operatorname{def} w - [\operatorname{def}\lambda : C : \operatorname{def}\lambda]). \tag{156}$$

The last representations show that the non-self-adjoint optimization problem is isomorphic to the problem of minimization of the energy stored in a dissipative media loaded by harmonic loading. Particularly, it allows us to apply methods of estimating the stored energy in an inhomogeneous media with complex-valued properties. We should mention that a new approach was recently suggested by Milton (1990b); Gibiansky and Milton (1993); Cherkaev and Gibiansky (1994), to establish the minimal variational principle for the problem. We could follow this approach here to establish the minimal variational principle, but we prefer to establish it in a more straightforward manner.

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